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Integral representations for the solutions of the quadratic pencil of the Sturm-Liouville equation with a discontinuous coefficient

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Abstract

In this study we construct new integral representations of Jost-type solutions of the quadratic pencil of the Sturm-Liouville equation with piece-wise constans coefficient on the entire axis under some boundness conditions of the potential functions.

Keywords: *Jost-type solutions, Sturm-Liouville pencil, discontinious Sturm-Liouville equation, integral equation, transformation operatör.*

1. INTRODUCTION

In the present study the Sturm-Liouville equation

$$-y'' + q(x)y + 2\lambda p(x)y = \lambda^2 \rho(x)y, x \in I = (-\infty, +\infty) \quad (1)$$

is considered where

$$\rho(x) = \begin{cases} 1, & x \geq 0, \\ \alpha^2, & x < 0, \end{cases} (\alpha \neq 1, \alpha > 0) \quad (2)$$

is the piece-wise constant coefficient, $\lambda$is a complex parameter, $q(x)$ and $p(x)$ are real functions such that

$$(1 + |x|)q(x), p(x) \in L^1(I), p(x) \in BC(I) \quad (3)$$

Here $L^1(I)$ is the space of summable functions on I and $BC(I)$ is the class of functions that are bounded and continuous on I . Equation (1) is related to solving the inverse problem for the Klein-Gordon equation with a static potential and zero charge in quantum scattering theory [7]. Some scattering problems arising in the theory of transmission lines, the theory of electromagnetism, and the theory of elasticity are also reduced to equation (1) [12]. It is well known that transformation operators method is an important method in the inverse problems theory. V.A. Marchenko [1, 8] applied the transformation operators to the solution of the inverse problems for Sturm-Liouviile operator on a finite interval and on the half line. Transformation operators were also used in the study of Levitan, Gasymov [1], where they obtained necessary and sufficient conditions for recovering a Sturm-Liouville operator from its spectral characteristics. In the case of $\rho(x) = 1$, there are enough studies in the literature using transformation type operators, called integral representations of the special solutions, to solve direct and inverse scattering problems of equation (1) [3, 6, 7, 11] . Some problems with various statements related to inverse scattering problems for the discontinuous Sturm-Liouville equation have been considered in [2, 4, 5, 9]. The direct and inverse scattering problems for equation (1) with $p(x) = 0$ in various settings have been investigated [4, 5, 13] where new integral representations, similar transformations operators for the Jost solutions of the Sturm-Liouville equation, are obtained and applied to the investigation of the considered problems. In [10] direct and inverse scattering problems have been investigated for equation (1) with discontinuity conditions. In this study we construct new integral representations of Jost-type solutions of equation (1) on the entire axis under conditions (2) and (3).

2. INTEGRAL REPRESENTATION OF THE JOST SOLUTIONS

We denote by $f_{\pm}(x,\lambda)$ the solution of (1) with the condition

$$\lim_{x\to\pm\infty} f_{\pm}(x,\lambda)\, e^{\mp\lambda\mu(x)} = 1,$$

where $\mu(x) = x\sqrt{\rho(x)}$. The solutions $f_+(x,\lambda)$ and $f_-(x,\lambda)$ will be called the right and the left Jost solutions of (1) respectively. It is easy to verify that the solution $f_{\pm}(x,\lambda)$ obeys the integral equation

$$f_{\pm}(x,\lambda) = e_{\pm}(x,\lambda) + \int_x^{\pm\infty} \left[\frac{1}{2}\left(\frac{1}{\sqrt{\rho(t)}} - \frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda(\mu(t)+\mu(x))}{\lambda} + \right. \tag{4}$$

$$\left. + \frac{1}{2}\left(\frac{1}{\sqrt{\rho(t)}} - \frac{1}{\sqrt{\rho(x)}}\right)\frac{\sin\lambda(\mu(t)-\mu(x))}{\lambda}\right] \mathrm{x} (q(t) + 2\lambda p(t) f_{\pm}(t,\lambda))dt,$$

where

$$e_+(x,\lambda) = \frac{1}{2}\left(1 + \frac{1}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu(x)} + \frac{1}{2}\left(1 - \frac{1}{\sqrt{\rho(x)}}\right)e^{-i\lambda\mu(x)}$$

and

$$e_-(x,\lambda) = \frac{1}{2}\left(1 - \frac{\alpha}{\sqrt{\rho(x)}}\right)e^{i\lambda\mu(x)} + \frac{1}{2}\left(1 + \frac{\alpha}{\sqrt{\rho(x)}}\right)e^{-i\lambda\mu(x)}$$

Consider the solution $f_+(x,\lambda)$ When $x > 0$ it is well known that [3,6] for all $Im\lambda \geq 0$ the solution $f_+(x,\lambda)$ has the representation

$$f_+(x,\lambda) = e^{i\lambda x + i\omega_+(x)} + \int_x^{+\infty} A^+(x,t)e^{i\lambda t}dt \tag{5}$$

where $\omega_+(x) = \int_x^{+\infty} p(t)dt$ and the kernel function $A^+(x,t)$ satisfies

$$\int_x^{+\infty} |A^+(x,t)| \leq C_0\sigma^+(x)e^{\sigma^+(x)} \tag{6}$$

for some constant $C_0 > 0$ and

$$\sigma^+(x) = \int_x^{+\infty} \left((1+t)|q(t)| + 2|p(t)|\right).$$

Moreover, the kernel function $A^+(x,t)$ satisfies the condition

$$A^+(x,x) = \frac{1}{2}\left(\int_x^{+\infty} [q(t) + p^2(t)]dt - ip(x)\right)e^{i\omega_+(x)}. \tag{7}$$

Consider the case $x < 0$ for the solution $f_+(x,\lambda)$ In this case the equation (4) takes the form of

$$f_+(x,\lambda)=\alpha_+e^{i\alpha\lambda x}+\alpha_-e^{-i\alpha\lambda x}+\int_x^0\frac{sin\alpha\lambda(t-x)}{\lambda}q(t)f_+(t,\lambda)dt+$$
$$+\int_0^\infty\left[\alpha^-\frac{sin\lambda(t+\alpha x)}{\lambda}+\alpha^+\frac{sin\lambda(t-\alpha x)}{\lambda}\right][q(t)+2\lambda p(t)]f_+(t,\lambda)dt, \quad (8)$$

where $\alpha_\pm=\frac{1}{2}(1\pm\frac{1}{\alpha})$. We require that the solution of the integral equation (8) has the form of

$$f_+(x,\lambda)=R_+(x)e^{i\alpha\lambda x}+R_-(x)e^{-i\alpha\lambda x}+\int_{\alpha x}^{+\infty}B^+(x,t)e^{i\lambda t}dt,\qquad Im\lambda\geq 0, x<0 \quad (9)$$

where

$$R_\pm(x)=\alpha_\pm e^{i\omega_+(0)\pm\frac{i}{\alpha}\int_x^0 p(t)dt}$$

and $B^+(x,t)$ is defined after replacing $f_+(x,\lambda)$ in equation (8) with formulas (5), (9) and transforming some integrals of the Fourier type:

$$B^+(x,t)=\frac{1}{2\alpha}\int_{\frac{t+\alpha x}{2\alpha}}^0 q(s)R_+(s)ds+\frac{1}{2\alpha}\int_{\frac{\alpha x-t}{2\alpha}}^0 q(s)R_-(s)ds-$$
$$-\frac{i}{2\alpha^2}p\left(\frac{t+\alpha x}{2\alpha}\right)R_+\left(\frac{t+\alpha x}{2\alpha}\right)+\frac{i}{2\alpha^2}p\left(\frac{\alpha x-t}{2\alpha}\right)R_-\left(\frac{\alpha x-t}{2\alpha}\right)+$$
$$+\frac{\alpha_+}{2}\int_0^{+\infty}q(s)\,e^{i\omega_+(s)}ds-\frac{\alpha_-}{2}\int_0^{\frac{t-\alpha x}{2}}q(s)\,e^{i\omega_+(s)}ds-\frac{i\alpha_-}{2}p\left(\frac{t-\alpha x}{2}\right)e^{i\omega_+\left(\frac{t-\alpha x}{2}\right)}+$$
$$+\alpha_+\int_0^{\frac{t-\alpha x}{2}}dv\int_v^{+\infty}q(u-v)A^+(u-v,u+v)du-$$
$$-\alpha_-\int_0^{\frac{t-\alpha x}{2}}dv\int_v^{\frac{t-\alpha x}{2}}q(u-v)A^+(u-v,u+v)du+$$
$$+\frac{1}{\alpha^2}\int_0^{\frac{t-\alpha x}{2}}dv\int_{\frac{t+\alpha x}{2}}^{v}q\left(\frac{u-v}{\alpha}\right)B^+\left(\frac{u-v}{\alpha},u+v\right)du+$$
$$+i\alpha_+\int_{\frac{t-\alpha x}{2}}^{+\infty}p\left(u-\frac{t-\alpha x}{2}\right)A^+\left(u-\frac{t-\alpha x}{2},u+\frac{t-\alpha x}{2}\right)du-$$

$$-i\alpha_- \int_0^{\frac{t-\alpha x}{2}} p\left(\frac{t-\alpha x}{2}-v\right) A^+\left(\frac{t-\alpha x}{2}-v, \frac{t-\alpha x}{2}+v\right) dv +$$

$$+\frac{1}{i\alpha^2} \int_0^{\frac{t-\alpha x}{2}} p\left(\frac{t+\alpha x}{2}-\frac{v}{\alpha}\right) B^+\left(\frac{t+\alpha x}{2\alpha}-\frac{v}{\alpha}, \frac{t+\alpha x}{2}+v\right) dv -$$

$$-\frac{1}{i\alpha^2} \int_0^{\frac{t-\alpha x}{2}} p\left(\frac{u}{\alpha}-\frac{t-\alpha x}{2\alpha}\right) B^+\left(\frac{u}{\alpha}-\frac{t-\alpha x}{2\alpha}, \frac{u}{\alpha}+\frac{t+\alpha x}{2}\right) du, -\alpha x \le t < -\alpha x \qquad (10)$$

$$B^+(x,t) = \frac{\alpha_+}{2} \int_{\frac{t+\alpha x}{2}}^{+\infty} q(s)ds + \frac{\alpha_-}{2} \int_{\frac{t-\alpha x}{2}}^{+\infty} q(s)ds -$$

$$-\frac{i\alpha_-}{2} p\left(\frac{t-\alpha x}{2}\right) e^{i\omega_+\left(\frac{t-\alpha x}{2}\right)} - \frac{i\alpha_+}{2} p\left(\frac{t+\alpha x}{2}\right) e^{i\omega_+\left(\frac{t+\alpha x}{2}\right)}$$

$$\alpha_+ \int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} dv \int_v^{+\infty} q(u-v) A^+(u-v, u+v) du +$$

$$+\alpha_+ \int_0^{\frac{t-\alpha x}{2}} dv \int_{\frac{t+\alpha x}{2}}^{+\infty} q(u-v) A^+(u-v, u+v) du -$$

$$-\alpha_- \int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} dv \int_v^{\frac{t-\alpha x}{2}} q(u-v) A^+(u-v, u+v) du +$$

$$+\frac{1}{\alpha^2} \int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} dv \int_{\frac{t+\alpha x}{2}}^{v} q\left(\frac{u-v}{\alpha}\right) B^+\left(\frac{u-v}{\alpha}, u+v\right) dv +$$

$$+i\alpha_+ \int_{\frac{t-\alpha x}{2}}^{+\infty} p\left(u-\frac{t-\alpha x}{2}\right) A^+\left(u-\frac{t-\alpha x}{2}, u+\frac{t-\alpha x}{2}\right) du -$$

$$-i\alpha_+ \int_0^{\frac{t+\alpha x}{2}} p\left(\frac{t+\alpha x}{2}-v\right) A^+\left(\frac{t+\alpha x}{2}-v, \frac{t+\alpha x}{2}+v\right) dv +$$

$$-i\alpha_- \int_0^{\frac{t-\alpha x}{2}} p\left(\frac{t-\alpha x}{2} - v\right) A^+\left(\frac{t-\alpha x}{2} - v, \frac{t-\alpha x}{2} + v\right) dv +$$

$$+i\alpha_- \int_{\frac{t+\alpha x}{2}}^{+\infty} p\left(u - \frac{t+\alpha x}{2}\right) A^+\left(u - \frac{t+\alpha x}{2}, u + \frac{t+\alpha x}{2}\right) du +$$

$$+\frac{1}{i\alpha^2} \int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} p\left(\frac{t+\alpha x}{2\alpha} - \frac{v}{\alpha}\right) B^+\left(\frac{t+\alpha x}{2\alpha} - \frac{v}{\alpha}, \frac{t+\alpha x}{2} + v\right) dv -$$

$$-\frac{1}{i\alpha^2} \int_{\frac{t+\alpha x}{2}}^{\frac{t-\alpha x}{2}} p\left(\frac{u}{\alpha} - \frac{t-\alpha x}{2\alpha}\right) B^+\left(\frac{u}{\alpha} - \frac{t-\alpha x}{2\alpha}, u + \frac{t-\alpha x}{2}\right) du, t > -\alpha x. \quad (11)$$

Here we suppose $A^+(x,t) \equiv 0$ for $t < 0$ and $B^+(x,t) \equiv 0$ for $t < \alpha x$. From equation (11) it is obtained that

$$\int_{\alpha x}^{+\infty} |B^+(x,t)|\, dt \le C\sigma^+(x) e^{\sigma^+(x)} \quad (11')$$

where $C > 0$ and

$$\sigma^+(x) = \frac{1}{2\alpha} \int_{\alpha x}^{+\infty} \left((1+t)|q(t)| + \frac{2}{\alpha}|p(t)| \right) dt.$$

By the similar way, considering the solution $f_-(x,\lambda)$ we have for $x < 0$

$$f_-(x,\lambda) = e^{-i\alpha\lambda x + i\omega_-(x)} + \int_{-\infty}^{\alpha x} A^-(x,t) e^{-i\lambda t} dt \ (Jm\lambda \ge 0), \quad (12)$$

where

$$\omega_-(x) = \frac{1}{\alpha} \int_{-\infty}^{x} p(t) dt$$

and the kernel function $A^-(x,t)$ satisfies the integral equation

$$A^-(x,t) = \frac{1}{2\alpha} \int_{-\infty}^{\frac{\alpha x + t}{2\alpha}} q(s) ds + \frac{1}{2i\alpha^2} p\left(\frac{t+\alpha x}{2\alpha}\right) e^{i\omega_-\left(\frac{t+\alpha x}{2}\right)} +$$

$$+\int_{-\infty}^{\frac{\alpha x+t}{2\alpha}} du \int_{-\infty}^{\frac{\alpha x - t}{2\alpha}} q(u+v) A^+\big(u+v, \alpha(u-v)\big) dv +$$

$$+\frac{1}{i\alpha} \int_0^{\frac{\alpha x - t}{2\alpha}} p\left(\frac{t+\alpha x}{2\alpha} + v\right) A^-\left(\frac{t+\alpha x}{2} + v, \alpha\left(\frac{t+\alpha x}{2} - v\right)\right) dv -$$

$$-\frac{1}{i\alpha}\int_{-\infty}^{\frac{\alpha x+t}{2\alpha}} p\left(u+\frac{\alpha x-t}{2\alpha}\right)A^-\left(u+\frac{\alpha x-t}{2\alpha},\alpha(u-\frac{\alpha x-t}{2\alpha})\right)du \quad (13)$$

which implies

$$\int_{-\infty}^{\alpha x} |A^-(x,t)|\,dt \le C_1\sigma^-(x)e^{\sigma^-(x)} \quad (14)$$

for some constant $C_1 > 0$ and

$$\sigma^-(x) = \frac{1}{2\alpha}\int_{-\infty}^{\alpha x}\left((1+t)|q(t)| + \frac{2}{\alpha}|p(t)|\right)dt.$$

Here $A^-(x,t) \equiv 0$ for $t > \alpha x$. Moreover, the kernel function $A^-(x,t)$ satisfies the condition

$$A^-(x,\alpha x) = \frac{1}{2\alpha}\left(\int_{-\infty}^{x}\left[q(t)+\frac{1}{\alpha^2}p^2(t)\right]dt + \frac{1}{2i\alpha}p(x)\right)e^{\omega_-(x)} \quad (15)$$

As in the case of the right Jost solution we have for $x > 0$

$$f_-(x,\lambda) = T_+(x)e^{i\lambda(x)} + T_-(x)e^{-i\lambda(x)} + \int_{-\infty}^{x} B^-(x,t)e^{-i\lambda t}dt\,, Im\lambda \ge 0, x > 0 \quad (16)$$

where

$$T_\pm(x) = \frac{1}{2}(1\mp\alpha)e^{i\omega_-(0)\mp i\int_0^x p(t)\,dt}$$

and

$$B^-(x,t) = \frac{\alpha_-}{2}\int_{\frac{t-x}{2\alpha}}^{0} q(s)e^{-i\omega_-(s)}\,ds + \frac{\alpha_+}{2}\int_{-\infty}^{0} q(s)e^{i\omega_-(s)}\,ds + \frac{1}{2}\int_{0}^{\frac{x+t}{2}} q(s)T_+(s)ds +$$

$$+\frac{1}{2}\int_{0}^{\frac{x-t}{2}} q(s)T_-(s)ds + \frac{i\alpha_-}{2\alpha}p\left(\frac{t-x}{2\alpha}\right)e^{i\omega_-\left(\frac{t-x}{2\alpha}\right)} - \frac{i}{2}p\left(\frac{t+x}{2}\right)T_-\left(\frac{t+x}{2}\right) +$$

$$+\frac{i}{2}p\left(\frac{x-t}{2}\right)T_+\left(\frac{x-t}{2}\right) + \alpha\alpha_-\int_{0}^{\frac{x-t}{2\alpha}} dv \int_{\frac{t-x}{2\alpha}}^{-v} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+\alpha\alpha_+\int_{0}^{\frac{x-t}{2\alpha}} dv \int_{-\infty}^{-v} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+\int_{0}^{\frac{x-t}{2}} dv \int_{-v}^{\frac{x+t}{2}} q(u+v)B^-(u+v,u-v)du +$$

$$+i\alpha_- \int_0^{\frac{x-t}{2\alpha}} p(\frac{t-x}{2\alpha}+v)A^-(\frac{t-x}{2\alpha}+v,\alpha\left(\frac{t-x}{2\alpha}-v\right)dv +$$

$$+i\alpha_+ \int_{-\infty}^{\frac{x-t}{2\alpha}} p(u-\frac{t-x}{2\alpha})A^-(u-\frac{t-x}{2\alpha},\alpha\left(\frac{t-x}{2\alpha}+u\right)du +$$

$$+i \int_{-\frac{x-t}{2}}^{\frac{x+t}{2}} p(u+\frac{x-t}{2})B^-(u+\frac{x-t}{2},\alpha\left(u-\frac{x-t}{2}\right)dv -$$

$$-i \int_0^{\frac{x-t}{2}} p\left(\frac{x+t}{2}+v\right)B^-\left(\frac{x+t}{2}+v,\frac{x+t}{2}-v\right)dv, -x<t\leq x, x>0 \qquad (17)$$

$$B^-(x,t) = \frac{\alpha_+}{2}\int_{-\infty}^{\frac{t+x}{2\alpha}} q(s)e^{-i\omega_-(s)}\,ds - \frac{\alpha_-}{2}\int_{-\infty}^{\frac{t-x}{2\alpha}} q(s)e^{i\omega_-(s)}\,ds +$$

$$\frac{i\alpha_-}{2\alpha}p\left(\frac{t-x}{2\alpha}\right)e^{i\omega_-\left(\frac{t-x}{2\alpha}\right)} - \frac{i\alpha_+}{2\alpha}p\left(\frac{t+x}{2\alpha}\right)e^{i\omega_-\left(\frac{t-x}{2\alpha}\right)} +$$

$$+\alpha_-\alpha \int_{-\frac{x+t}{2\alpha}}^{\frac{x-t}{2\alpha}} dv \int_{-\frac{x-t}{2\alpha}}^{-v} q(u+v)A^-(u+v,u-v)du -$$

$$-\alpha_-\alpha \int_0^{-\frac{x+t}{2\alpha}} dv \int_{-\infty}^{\frac{t-x}{2\alpha}} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+\alpha\alpha_+ \int_0^{-\frac{x+t}{2\alpha}} dv \int_{-\infty}^{\frac{t-x}{2\alpha}} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+\alpha\alpha_+ \int_{-\frac{x+t}{2\alpha}}^{\frac{x-t}{2\alpha}} dv \int_{-\infty}^{-v} q(u+v)A^-\big(u+v,\alpha(u-v)\big)du +$$

$$+ \int_{-\frac{x+t}{2\alpha}}^{\frac{x-t}{2}} dv \int_{-v}^{\frac{x+t}{2}} q(u+v)B^-(u+v,u-v)du -$$

$$-i\alpha_- \int_{-\infty}^{\frac{t+x}{2\alpha}} p\left(u-\frac{t+x}{2\alpha}\right)A^-\left(u-\frac{t+x}{2\alpha},u+\frac{t+x}{2\alpha}\right)du+$$

$$+i\alpha_- \int_{0}^{\frac{x-t}{2\alpha}} p\left(\frac{t-x}{2\alpha}+v\right)A^-\left(\frac{t-x}{2\alpha}+v,\alpha\left(\frac{t-x}{2}-v\right)\right)dv+$$

$$-i\alpha_+ \int_{-\infty}^{\frac{t-x}{2\alpha}} p\left(u+\frac{x-t}{2\alpha}\right)A^-\left(u+\frac{x-t}{2\alpha},\alpha\left(u-\frac{x-t}{2\alpha}\right)\right)du-$$

$$-i\alpha_+ \int_{0}^{-\frac{x+t}{2\alpha}} p\left(\frac{x+t}{2\alpha}+v\right)A^-\left(\frac{x+t}{2\alpha},\alpha\left(\frac{x+t}{2\alpha}-v\right)\right)du+$$

$$+i\int_{\frac{t-x}{2}}^{\frac{t+x}{2}} p\left(u-\frac{t-x}{2}\right)B^-\left(u-\frac{t-x}{2},u+\frac{t-x}{2}\right)du- \tag{18}$$

$$-i\int_{0}^{\frac{t-x}{2}} p\left(\frac{x+t}{2}-v\right)B^-\left(\frac{x+t}{2}-v,\frac{x+t}{2}+v\right)dv,\ \ t<-x<0$$

and $B^-(x,t)\equiv 0$ for $t>x$. Estimating (10), (11) and (17), (18) we can easily obtain that

$$\int_{-\infty}^{0}|B^-(x,t)|\le C_1\sigma^-(x)e^{\sigma^-(x)} \tag{19}$$

for some $C_1>0$ where

$$\sigma^-(x)=\left(\int_{-\infty}^{x}((1+t)\,|q(t)|+2|p(t)|)\right)dt$$

Hence by setting $K^\pm(x,t)=\begin{cases}A^\pm(x,t), & \pm x\ge 0\\ B^\pm(x,t), & \pm x<0\end{cases}$ and combining all our results we have the following theorem.

Theorem 1. For all $Im\lambda\ge 0$ *the solutions* $f_+(x,\lambda)$ *and* $f_-(x,\lambda)$ *can be respresented as*

$$f_+(x,\lambda)=R_+(x)e^{i\lambda\mu(x)}+R_-(x)e^{-i\lambda\mu(x)}+\int_{\mu(x)}^{+\infty}K^+(x,t)e^{i\lambda t}dt$$

and

$$f_-(x,\lambda)=T_+(x)e^{i\lambda\mu(x)}+T_-(x)e^{-i\lambda\mu(x)}+\int_{-\infty}^{\mu(x)}K^-(x,t)e^{-i\lambda t}dt$$

respectively, where $\mu(x)=x\sqrt{\rho(x)}$,

$$R_+(x)=\frac{1}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)e^{i\int_x^{+\infty}\frac{\rho(t)}{\sqrt{\rho(t)}}dt},$$

$$R_-(x)=\frac{1}{2}(1-\frac{1}{\sqrt{\rho(x)}})e^{i\int_x^{+\infty}\frac{\rho(t)sgnt}{\sqrt{\rho(t)}}dt},$$

$$T_+(x)=\frac{1}{2}(1-\frac{\alpha}{\sqrt{\rho(x)}})e^{-i\int_{-\infty}^{x}\frac{\rho(t)sgnt}{\sqrt{\rho(t)}}dt},$$

$$T_-(x)=\frac{1}{2}(1+\frac{\alpha}{\sqrt{\rho(x)}})e^{i\int_{-\infty}^{x}\frac{\rho(t)}{\sqrt{\rho(t)}}dt},$$

From the estimations (6), (11′), (14) and (19) we have

$$\int_{\mu(x)}^{+\infty}|K^+(x,t)|dt\leq C\sigma^+(\mu(x))e^{\sigma^+(\mu(x))} \tag{20}$$

and

$$\int_{-\infty}^{\mu(x)}|K^-(x,t)|dt\leq C\sigma^-(\mu(x))e^{\sigma^-(\mu(x))} \tag{21}$$

for some $C>0$. Now let

$$G^+(x,t)=-\int_t^{+\infty}K^+(x,s)ds-R^-(x)H(-\mu(x)-t),t\geq\mu(x),$$

$$G^-(x,t)=\int_{-\infty}^{t}K^+(x,s)ds+T_+(x)H(\mu(x)+t),t\leq\mu(x). \tag{22}$$

Where

$$H(y)=\begin{cases}1, & y>0\\ 0, & y<0\end{cases}$$

Then by using integration in parts we have

$$f_+(x,\lambda)=R_+(x)e^{i\lambda\mu(x)}+R_-(x)e^{-i\lambda\mu(x)}+\int_{\mu(x)}^{+\infty}K^+(x,t)e^{i\lambda t}dt=$$

$$=R_+(x)e^{i\lambda\mu(x)}+R_-(x)e^{-i\lambda\mu(x)}+$$

$$+e^{i\lambda\mu(x)}\int_{\mu(x)}^{+\infty}K^+(x,s)ds+i\lambda\int_{\mu(x)}^{+\infty}\left(\int_t^{+\infty}K^+(x,s)ds\right)e^{i\lambda t}dt=$$

$$=-f_+(x,0)e^{i\lambda\mu(x)}+i\lambda R_-(x)\int_{\mu(x)}^{-\mu(x)}e^{i\lambda t}dt+i\lambda\int_{\mu(x)}^{+\infty}\left(\int_t^{+\infty}K^+(x,s)ds\right)e^{i\lambda t}dt=$$

$$=f_+(x,0)e^{i\lambda\mu(x)}-i\lambda\int_{\mu(x)}^{+\infty}G^+(x,t)e^{i\lambda t}dt,$$

that is

$$f_+(x,\lambda)=f_+(x,0)e^{i\lambda\mu(x)}-i\lambda\int_{\mu(x)}^{+\infty}G^+(x,t)e^{i\lambda t}dt. \tag{23}$$

Similarly,

$$f_-(x,\lambda) = T_+(x)e^{i\lambda\mu(x)} + T_-(x)e^{-i\lambda\mu(x)} + \int_{-\infty}^{\mu(x)} K^-(x,t)e^{-i\lambda t}dt =$$

$$= T_+(x)e^{i\lambda\mu(x)} + T_-(x)e^{-i\lambda\mu(x)} - \int_{-\infty}^{x} K^-(x,s)ds +$$

$$+i\lambda \int_{-\infty}^{\mu(x)} \left(\int_{-\infty}^{t} K^-(x,s)ds \right) e^{-i\lambda t}dt = f_-(x,0)e^{-i\lambda\mu(x)} +$$

$$+i\lambda T_+(x) \int_{-\mu(x)}^{\mu(x)} e^{-i\lambda t}dt + i\lambda \int_{-\infty}^{\mu(x)} \left(\int_{-\infty}^{t} K^-(x,s)ds \right) e^{-i\lambda t}dt =$$

$$= f_-(x,0)e^{-i\lambda\mu(x)} + i\lambda \int_{-\infty}^{\mu(x)} G^-(x,t)e^{-i\lambda t}dt,$$

so we get

$$f_-(x,\lambda) = f_-(x,0)e^{-i\lambda\mu(x)} + i\lambda \int_{-\infty}^{\mu(x)} G^-(x,t)e^{-i\lambda t}dt \qquad (24)$$

.

Therefore the following theorem is true:

Theorem 2. *If $(1+|x|)q(x), p(x) \in L^1(I)$ then the Jost solutions $f_+(x,\lambda)$ and $f_-(x,\lambda)$ are expressed as (21) and (22) respectively, where the kernels $G^{\pm}(x,t)$ are continous and bounded functions for $x \in I$, $\pm t \geq \pm\mu(x)$. Additionally, if $p(x) \in BC(I)$ then $G_t{}^{\pm}(x,t)$ are continous and bounded for $\pm\mu(x) \leq \pm t$ as well as $G_t{}^{+}(x,t) \in L_1(\mu(x),+\infty)$ and $G_t{}^{-}(x,t) \in L_1(-\infty,\mu(x))$. Moreover the following relations are satisfied:*

$$f_+(x,0) = G^+(x,\mu(x)) = R_+(x),\ f_-(x,0) = G^-(x,\mu(x)) = T_-(x),$$

$$G_t{}^+(x,\mu(x)) = R_+(x)\left\{\frac{1}{2}\int_x^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds - \right.$$
$$\left. -\frac{i}{2}\frac{p(x)}{\rho(x)} - \frac{i}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)^2 p(0)\right\},$$

$$G_t{}^+(x,-\mu(x)+0) - G_t{}^+(x,-\mu(x)-0) = R_-(x)\left\{\frac{1}{2}\int_x^{+\infty}\left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgns ds - \right.$$
$$\left. \frac{i}{2}\frac{p(x)}{\rho(x)} + \frac{i}{2}\left(1+\frac{1}{\sqrt{\rho(x)}}\right)^2 p(0)\right\},$$

$$G_t{}^-(x,\mu(x)) = T_-(x)\left\{\frac{1}{2}\int_{-\infty}^{x}\left(\frac{q(s)}{\sqrt{\rho(s)}} + \frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)ds - \right.$$
$$\left. -\frac{i}{2}\frac{p(x)}{\rho(x)} - \frac{i}{2}\left(1-\frac{1}{\sqrt{\rho(x)}}\right)^2 p(0)\right\},$$

$$G_t^{-}(x,-\mu(x)+0)-G_t^{-}(x,-\mu(x)-0)=$$
$$=T_+(x)\left\{\frac{1}{2}\int_{-\infty}^{x}\left(\frac{q(s)}{\sqrt{\rho(s)}}+\frac{p^2(s)}{\rho(s)\sqrt{\rho(s)}}\right)sgnsds+\right.$$
$$\left.+\frac{i}{2}\frac{p(x)}{\rho(x)}+\frac{i}{2}\left(1+\frac{\sqrt{\rho(x)}}{\alpha}\right)^2p(0)\right\}.$$

Declarations
Conflict of interest The Authors declare that they have no conflicts of interest.
Ethical approval This article does not contain any studies with human participants or animals performed by any of the authors.

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